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Gauge and Bäcklund Transformations for the Generalized Sine-Gordon Equation and Its η-Dependent Modified Equation

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We study the generalized sine-Gordon hierarchy and its associated η-dependent modified sine-Gordon hierarchy. Two Bäcklund transformations for these two families are constructed. One of them is a generalization of the Bäcklund transformations of Wadati *et al.* and the other one is new. Gauge transformations of a relevant AKNS system are employed to reduce the integration of these equations via the Bäcklund transformations to quadratures. Three generations of explicit solutions of the sine-Gordon equation are presented.

1. INTRODUCTION

In a previous work (Zheng and Chan, 1988) we developed a gauge-Bäcklund transformation technique for constructing families of solutions to the hierarchy of the Korteweg–de Vries equation (KdVE). This method was extended to the case for the hierarchy of the modified KdVE (Zheng and Chan, in press). In this paper we extend the method further to include the generalized sine-Gordon hierarchy:

$$z_{xt} = \{(\cos z\, I \cos z + \sin z\, I \sin z) I\}^m \sin z, \qquad m \geq 1 \tag{1.1}$$

$$z_{xt} = \sin z, \qquad m = 0 \tag{1.2}$$

in which I is the integration operator.

This hierarchy was introduced in Sasaki and Bullough (1981), where polynomial and nonlocal conserved Hamiltonian densities were deduced from a geometric approach. Here we derive an expression for the equation of motion for an arbitrary member of the generalized sine-Gordon equation

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(GSGE) by a different method which leads us to the introduction of an associated family of nonlinear evolution equations: the η-dependent modified GSGE (η-mGSGE). The solutions of the two hierarchies are in one-to-one correspondence and the situation is analogous to the KdVE and η-mKdVE pair.

The paper is organized as follows. In Section 2 the GSGE is derived, and in Section 3 a Bäcklund transformation (BT) for it is established. In order to implement this BT to obtain explicit solutions, we consider in Section 4 gauge transformations (GT) of the relevant AKNS system in the spirit of our previous work (Zheng and Chan, 1988, and in press). Section 5 introduces the new η-mGSGE and its BT. Finally, in Section 6 we summarize our construction procedures, and three generations of solutions for the sine-Gordon equation are presented to illustrate how the integration of this equation is reduced to quadratures.

2. GENERALIZED SINE-GORDON EQUATION

It is well known that the SGE (1.2), as a condition of integrability, can be derived from the following AKNS system (Ablowitz and Segur, 1981):

$$d\Psi = \Omega\Psi \tag{2.1}$$

where Ψ is a column vector function of x and t,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.2}$$

and

$$\Omega = P\,dx + Q\,dt \tag{2.3}$$

$$P = \begin{pmatrix} \eta & q \\ -q & -\eta \end{pmatrix} \tag{2.4}$$

η is a real parameter, independent of x and t (2.5)

q is a real function of x and t (2.6)

$$Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2.7}$$

A is a functional of q (2.8)

$$B = \frac{A_x}{2q} + \frac{1}{\eta} qA + \frac{1}{4\eta}\left(\frac{A_x}{q}\right)_x \tag{2.9}$$

$$C = \frac{A_x}{2q} - \frac{1}{\eta} qA - \frac{1}{4\eta}\left(\frac{A_x}{q}\right)_x \tag{2.10}$$

First we cite some useful results about this system (Zheng and Chan, to appear).

(C1) A necessary and sufficient condition for the integrability of the AKNS system (2.1) is that A and q satisfy the following equation:

$$q_t+\eta\frac{A_x}{q}-\frac{1}{\eta}(qA)_x-\frac{1}{4\eta}\left(\frac{A_x}{q}\right)_{xx}=0 \tag{2.11}$$

(C2) There exists a complex gauge

$$G_1=\begin{pmatrix} iq-2\eta & -q \\ 1 & i \end{pmatrix} \tag{2.12}$$

which carries the following transformation:

$$G_1:\quad \Psi\to\Phi=G_1\Psi \tag{2.13}$$

with

$$\Phi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{pmatrix}-2\eta\psi_1-q\psi_2+iq\psi_1\\ \psi_1+i\psi_2\end{pmatrix} \tag{2.14}$$

and Φ satisfies a complex AKNS system:

$$\begin{aligned} d\Phi=&\begin{pmatrix}\eta & u\\ -1 & -\eta\end{pmatrix}\Phi\,dx\\ &+\begin{pmatrix}-\frac{1}{2}\hat{C}_x-\eta\hat{C} & -\frac{1}{2}\hat{C}_{xx}-\eta\hat{C}_x-u\hat{C}\\ \hat{C} & \frac{1}{2}\hat{C}_x+\eta\hat{C}\end{pmatrix}\Phi\,dt \end{aligned} \tag{2.15}$$

where

$$u=iq_x+q^2 \qquad \text{(the complex Muira transformation)} \tag{2.16}$$

$$\hat{C}=-\frac{1}{2\eta q}RA \tag{2.17}$$

$$R=iD+2q, \qquad D=\frac{\partial}{\partial x} \tag{2.18}$$

(C3) A necessary and sufficient condition for the integrability of the AKNS system (2.15) is that $\hat{C}$ and u satisfy the following equation:

$$u_t+\tfrac{1}{2}\hat{C}_{xxx}+2(u-\eta^2)\hat{C}_x+u_x\hat{C}=0 \tag{2.19}$$

(C4) Under the condition that u and q satisfy (2.16), the expressions on the left-hand side of equations (2.11) and (2.19) possess the following relationship:

$$\begin{aligned} &u_t+\tfrac{1}{2}\hat{C}_{xxx}+2(u-\eta^2)\hat{C}_x+u_x\hat{C}\\ &=R\left\{q_t+\eta\frac{A_x}{q}-\frac{1}{\eta}(qA)_x-\frac{1}{4\eta}\left(\frac{A_x}{q}\right)_{xx}\right\} \end{aligned} \tag{2.20}$$

This identity establishes a relation between the two equations (2.11) and (2.19).

Now we choose A in (2.11) to be a polynomial of η^{-1}:

$$A = \sum_{j=0}^{n} A_j \eta^{-2(n-j)-1} \tag{2.21}$$

Substituting (2.21) into (2.11) and equating to zero all of the coefficients of the power of η^{-1}, we get

$$\left[\frac{1}{4}\left(\frac{A_{0,x}}{q}\right)_x + qA_0\right]_x = 0 \tag{2.22}$$

$$\left[\frac{1}{4}\left(\frac{A_{j,x}}{q}\right)_x + qA_j\right]_x = \frac{A_{j-1,x}}{q}, \qquad j = 1, 2, \ldots, n \tag{2.23}$$

$$q_t + \frac{A_{n,x}}{q} = 0 \tag{2.24}$$

Denote

$$G = D(\tfrac{1}{4}D + qD^{-1}q) \tag{2.25}$$

Then (2.23) can be rewritten as

$$Gq^{-1}DA_j = q^{-1}DA_{j-1}, \qquad j = 1, 2, \ldots, n \tag{2.26}$$

Using the inverse operator G^{-1} of G in (2.25), we get the following recursion formula:

$$A_j = D^{-1}qG^{-1}q^{-1}DA_{j-1}, \qquad j = 1, 2, \ldots, n \tag{2.27}$$

Thus, to each solution A_0 of equation (2.22), we have

$$A_j = D^{-1}qG^{-j}q^{-1}DA_0, \qquad j = 1, 2, \ldots, n \tag{2.28}$$

Taking $j = n$ in (2.28) and substituting it into (2.24), we get the following evolution equation:

$$q_t + G^{-n}q^{-1}DA_0 = 0, \qquad n = 0, 1, 2, \ldots \tag{2.29}$$

This is a set of integrodifferential equations, since G^{-1} involves the integral operator. To obtain the functional A_0, we introduce a new function

$$z = 2\int q\,dx, \qquad \text{or} \qquad q = \tfrac{1}{2}z_x \tag{2.30}$$

and assume that

$$A_0 = A_0\left(2\int q\,dx\right) = A_0(z) \tag{2.31}$$

By (2.22), (2.30), and (2.31), we have

$$A_0 = \tfrac{1}{4}\cos z \tag{2.32}$$

Substituting (2.30) and (2.32) into (2.28) and (2.29) we get

$$A_j = -\tfrac{1}{4}D_z^{-1}G^{-j}\sin z, \qquad j = 1, 2, \ldots, n \tag{2.33}$$

$$z_{xt} - G^{-n}\sin z = 0, \qquad n = 0, 1, 2, \ldots \tag{2.34}$$

where

$$G = \tfrac{1}{4}Dz_x(D_z + D_z^{-1}) \tag{2.35}$$

$$D_z = \frac{\partial}{\partial z}, \qquad D_z^{-1} = \int \cdot\, dz \tag{2.36}$$

$$G^{-1} = (\cos z D_z^{-1}\cos z + \sin z D_z^{-1}\sin z)D_z^{-1}$$

(Sasaki and Bullough, 1981).

For $n = 0$, (2.34) reduces to the SGE (1.2); therefore we call (2.34) the generalized sine-Gordon equation (GSGE).

Inserting (2.33) into (2.21), (2.9), and (2.10) gives

$$A = -\frac{1}{4}\sum_{j=0}^{n} D_z^{-1}G^{-j}\sin z\,\eta^{-2(n-j)-1} \tag{2.37}$$

$$B = -\frac{1}{4}\sum_{j=0}^{n} G^{-j}\sin z\,\eta^{-2(n-j)-1} - \frac{1}{2}\sum_{j=0}^{n} D^{-1}G^{-j+1}\sin z\,\eta^{-2(n-j+1)} \tag{2.38}$$

$$C = -\frac{1}{4}\sum_{j=0}^{n} G^{-j}\sin z\,\eta^{-2(n-j)-1} + \frac{1}{2}\sum_{j=0}^{n} D^{-1}G^{-j+1}\sin z\,\eta^{-2(n-j+1)} \tag{2.39}$$

By the above results, we have the following.

Theorem 1. Under the condition that q and z be connected by (2.30) and A, B, and C take the values (2.37)–(2.39), a necessary and sufficient condition for the integrability of the AKNS system (2.1) is that z satisfies the GSGE (2.34).

3. BÄCKLUND TRANSFORMATION FOR THE GSGE

Wadati *et al.* (1975) constructed a BT for the SGE (1.2) which reads

$$z' = z + 4\tan^{-1}\frac{\psi_2}{\psi_1} \tag{3.1}$$

where ψ_1 and ψ_2 are solutions of the AKNS system (2.1)–(2.10) corresponding to the solution z of the SGE (1.1). We now show that (3.1) is also valid for the GSGE (2.34).

Substituting (2.28) into (2.21) gives

$$A = \sum_{j=0}^{n} D^{-1}qG^{-j}q^{-1}DA_0\eta^{-2(n-j)-1} \tag{3.2}$$

Denote

$$F = \tfrac{1}{4}D^2 + u + \tfrac{1}{2}u_xD^{-1} \tag{3.3}$$

It is easy to check that the commutative relations

$$Dq^{-1}R = Rq^{-1}D \tag{3.4}$$

$$RG = FR \qquad \text{or} \qquad RG^{-1} = F^{-1}R \tag{3.5}$$

hold. Substituting (3.2) into (2.17) and using (3.4) and (3.5), we have

$$\hat{C} = \sum_{j=0}^{n} D^{-1}F^{-j}DC_0\eta^{-2(n-j)-2} \tag{3.6}$$

where, by (2.32)

$$C_0 = -\tfrac{1}{2}q^{-1}RA_0 = -\tfrac{1}{4}e^{-iz} \tag{3.7}$$

By virtue of (2.22), (2.25), (3.5), and (3.4), we find that the C_0 in (3.7) satisfies the following equation:

$$FDC_0 = 0 \tag{3.8}$$

Now inserting (3.6) into (2.19) and using (3.8), we obtain the following equation:

$$u_t - 2F^{-n}DC_0 = 0 \tag{3.9}$$

By (2.30) and (2.16), z is in fact a functional of u, and so is C_0 by (3.7). Thus, by (3.3), we have the following result.

Theorem 2. A necessary and sufficient condition for the integrability of the AKNS system (2.15) with $\hat{C}$ in (3.6) is that the function u satisfies equation (3.9).

Referring to (2.19), (3.9), (2.11), (2.34), and (2.20), we get the following equality:

$$u_t - 2F^{-n}DC_0 = R(z_{xt} - G^{-n}\sin z) \tag{3.10}$$

Note that R is a complex operator; the equality (3.10) implies the following result.

Theorem 3. Under the condition that u and z are connected by (2.16) and (2.30), a necessary and sufficient condition of u satisfying equation (3.9) is that z satisfy equation (2.34).

Now, assume that z is a known solution of the GSGE (2.34), and ψ_1 and ψ_2 are the corresponding solutions of the AKNS system (2.1)-(2.7) with q, A, B, and C given in (2.30) and (2.37)-(2.39). Then, by (2.16), (2.17), and (2.14), we get u, $\hat{C}$, and Φ; they satisfy the AKNS system (2.15), or in component form

$$\varphi_{1x} = \eta\varphi_1 + u\varphi_2 \tag{3.11}$$

$$\varphi_{2x} = -\varphi_1 - \eta\varphi_2 \tag{3.12}$$

$$\varphi_{1t} = -(\tfrac{1}{2}\hat{C}_x + \eta\hat{C})\varphi_1 - (\tfrac{1}{2}\hat{C}_{xx} + \eta\hat{C}_x + u\hat{C})\varphi_2 \tag{3.13}$$

$$\varphi_{2t} = \hat{C}\varphi_1 + (\tfrac{1}{2}\hat{C}_x + \eta\hat{C})\varphi_2 \tag{3.14}$$

Define

$$v = -\frac{\varphi_1}{\varphi_2} - \eta = \frac{\varphi_{2x}}{\varphi_2} \tag{3.15}$$

By (3.15) and (3.11)-(3.14), we have

$$u = \eta^2 - v_x - v^2 \tag{3.16}$$

$$v_t = (\tfrac{1}{2}\hat{C}_x - v\hat{C})_x \tag{3.17}$$

Denote

$$R^+ = D - 2v \tag{3.18}$$

$$R^- = -D - 2v \tag{3.19}$$

Then (3.16) gives

$$u_x = R^- v_x \tag{3.20}$$

$$u_t = R^- v_t \tag{3.21}$$

Substituting (3.6) into (3.17) and using (3.18), we get

$$v_t = \tfrac{1}{2}\sum_{j=0}^{n} DR^+D^{-1}F^{-j}DC_0\eta^{-2(n-j)-2} \tag{3.22}$$

From (3.9) and (3.21) we have

$$DC_0 = \tfrac{1}{2}F^n R^- v_t \tag{3.23}$$

Substituting (3.23) into (3.22) to eliminate DC_0, we obtain

$$v_t = \frac{1}{4}\left(\sum_{j=0}^{n} DR^+ D^{-1} F^{n-j} R^- \eta^{-2(n-j)-2}\right) v_t \tag{3.24}$$

Denote

$$S = \tfrac{1}{4}D^2 + \eta^2 - v^2 - v_x D^{-1} v \tag{3.25}$$

Then we have

$$FR^- = R^- S \tag{3.26}$$

$$DR^+ D^{-1} R^- = -4(S - \eta^2) \tag{3.27}$$

Applying (3.25)-(3.27) to (3.24), we obtain an evolution equation in compact form:

$$S^{n+1} v_t = 0 \tag{3.28}$$

Thus we arrive at the following result.

Theorem 4. Let u be a solution of equation (3.9), and φ_1 and φ_2 be the corresponding solutions of (3.11)-(3.14); then the function v defined by (3.15) is a solution of equation (3.28).

Using the relation between u and v of (3.16) and by direct calculation, one finds that equations (2.19) and (3.17) are connected by the following relation:

$$u_t + \tfrac{1}{2}\hat{C}_{xxx} + 2(u - \eta^2)\hat{C}_x + u_x \hat{C} = R^-[v_t - (\tfrac{1}{2}\hat{C}_x - v\hat{C})_x] \tag{3.29}$$

This implies the following equality between equations (3.9) and (3.28):

$$u_t - 2F^{-n} DC_0 = R^-(S^{n+1} v_t) \tag{3.30}$$

We state this result in a theorem.

Theorem 5. Whenever v is a solution of equation (3.28), the function u determined by v in (3.16) is a solution of equation (3.9).

Note that the operator S defined in (3.25) is even with respect to v. Therefore, equation (3.28) is odd with respect to v. Thus, equation (3.28) possesses with every solution v another solution $-v$. But then, by substituting $-v$ into (3.16) and by Theorem 3, we obtain another solution u' of equation (3.9):

$$u' = \eta^2 + v_x - v^2 \tag{3.31}$$

Subtracting (3.6) from (3.31), we get

$$u' = u + 2v_x \tag{3.32}$$

This is a BT for equation (3.9). According to Theorem 3, we can expect that it contains a new solution of equation (2.34), provide the right-hand side of (3.32) possesses the form of a complex Miura transformation (2.16). We want to show indeed this is the case.

Denote

$$iq'_x + q^* = u' = u + 2v_x \tag{3.33}$$

where q' and q^* are two real functions. One needs to show that the following equality holds:

$$q^* = (q')^2 \tag{3.34}$$

Substituting the complex function φ_2 in (2.14) into (3.15) and then (2.16) and (3.15) into (3.33), we have

$$iq'_x + q^* = iq_x + q^2 + 2\left[\ln(\psi_1^2 + \psi_2^2)^{1/2} + i\tan^{-1}\frac{\psi_2}{\psi_1}\right]_{xx} \tag{3.35}$$

Equating the imaginary part and the real part of the two sides of equality (3.35), respectively, gives

$$q' = q + 2\left(\tan^{-1}\frac{\psi^2}{\psi_1}\right)_x \tag{3.36}$$

$$q^* = q^2 + 2[\ln(\psi_1^2 + \psi_2^2)^{1/2}]_{xx} \tag{3.37}$$

Using (2.1) and (2.4), we find that (3.36) and (3.37) are simplified to

$$q' = -q - \frac{4\eta\psi_1\psi_2}{\psi_1^2 + \psi_2^2} \tag{3.38}$$

$$q^* = \left(q + \frac{4\eta\psi_1\psi_2}{\psi_1^2 + \psi_2^2}\right)^2 \tag{3.39}$$

(3.38) and (3.39) indicate that equality (3.34) holds. Therefore (3.33) can be rewritten in the following form of a complex Miura transformation:

$$u' = iq'_x + q'^2 \tag{3.40}$$

where q' is the function defined in (3.36). Thus, by Theorem 3, the function

$$z' = 2\int q'\,dx \tag{3.41}$$

is a solution of the GSGE (2.34). Substituting (2.30) into (3.36) and then (3.36) into (3.41), we get (3.1). This means that (3.1) is a BT for the GSGE (2.34). We state this in the following result.

Theorem 6. Assume that z is a solution of the GSGE (2.34) and ψ_1 and ψ_2 are the corresponding solutions of the AKNS system (2.1)–(2.7) with q, A, B, and C given in (2.30) and (2.37)-(2.39); then the function z' defined in (3.1) is a new solution of the GSGE (2.34), that is, (3.1) is a BT for the GSGE (2.34).

4. GAUGE TRANSFORMATIONS FOR AKNS SYSTEMS

The application of the BT (3.1) for finding new solution of the GSGE (2.34) requires the solutions ψ_1 and ψ_2 of the AKNS system (2.1). In this section, we introduce an easy method to obtain a new solution of (2.1) from a known solution, that is, the gauge transformation (GT) method for the AKNS system (2.1).

To the solution z' in (3.1) of the GSGE (2.34), by Theorem 1, there is a corresponding integrable AKNS system

$$d\Psi' = \Omega'\Psi' \tag{4.1}$$

where Ψ' is a column vector function of x and t,

$$\Psi' = \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} \tag{4.2}$$

and

$$\Omega' = P'\,dx + Q'\,dt \tag{4.3}$$

$$P' = \begin{pmatrix} \eta & q' \\ -q' & -\eta \end{pmatrix} \tag{4.4}$$

$$\eta \text{ kept the same as (2.5)} \tag{4.5}$$

$$q' = \tfrac{1}{2} z'_x \tag{4.6}$$

$$Q' = \begin{pmatrix} A' & B' \\ C' & -A' \end{pmatrix} \tag{4.7}$$

$$A' = -\tfrac{1}{4} \sum_{j=0}^{n} D_z^{-1} G^{-j} \sin z'\, \eta^{-2(n-j)-1} \tag{4.8}$$

$$B' = -\tfrac{1}{4} \sum_{j=0}^{n} G^{-j} \sin z'\, \eta^{-2(n-j)-1} - \tfrac{1}{2} \sum_{j=0}^{n} D^{-1} G^{-j+1} \sin z'\, \eta^{-2(n-j+1)} \tag{4.9}$$

$$C' = -\tfrac{1}{4} \sum_{j=0}^{n} G^{-j} \sin z'\, \eta^{-2(n-j)-1} + \tfrac{1}{2} \sum_{j=0}^{n} D^{-1} G^{-j+1} \sin z'\, \eta^{-2(n-j+1)} \tag{4.10}$$

Referring to (2.13)-(2.19), there exists a complex GT

$$G_3: \quad \Psi' \to \Phi' = G_3\Psi' \tag{4.11}$$

with

$$G_3 = \begin{pmatrix} iq' - 2\eta & -q' \\ 1 & i \end{pmatrix} \tag{4.12}$$

and

$$\Phi' = \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} -2\eta\psi_1' - q\psi_2' + iq\psi_1' \\ \psi_1' + i\psi_2' \end{pmatrix} \tag{4.13}$$

such that Φ' satisfies the following complex AKNS system:

$$\begin{aligned} d\Phi' &= \begin{pmatrix} \eta & u' \\ -1 & -\eta \end{pmatrix} \Phi' \, dx \\ &+ \begin{pmatrix} -\frac{1}{2}\hat{C}_x' - \eta\hat{C}' & -\frac{1}{2}\hat{C}_{xx}' - \eta\hat{C}_x' - u\hat{C}' \\ \hat{C}' & \frac{1}{2}\hat{C}_x' + \eta\hat{C}' \end{pmatrix} \Phi' \, dt \end{aligned} \tag{4.14}$$

where u' is a solution of equation (3.9) defined in (3.31) or (3.32), and $\hat{C}'$ is connected to q' and A' as follows:

$$\hat{C}' = -\frac{1}{2\eta q'} R'A' \qquad \left(R' = i\frac{\partial}{\partial x} + 2q' \right) \tag{4.15}$$

Now we have two AKNS systems (2.15) and (4.14). In Zheng and Chan (1988) we point out that these two AKNS systems possess a GT under the condition that the function v defined in (3.15) satisfies an odd evolution equation. In the present case, this condition is also satisfied, namely, equation (3.28). Therefore, we can apply those results to (2.15) and (4.14). We cite them in the following.

There exists a GT

$$G_2: \quad \Phi \to \Phi' = G_2\Phi \tag{4.16}$$

which transforms (2.15) into (4.14), where G_2 is a 2×2 matrix as follows:

$$G_2 = \frac{1}{\beta^2} \begin{pmatrix} a + (\eta - v)c & (\eta + v)a - b + (\eta^2 - v^2)c - (\eta - v)d \\ -c & -(\eta + v)c + d \end{pmatrix} \tag{4.17}$$

The notations in (4.17) have the following meanings. η and v have been defined in (2.5) and (3.15). Denote

$$\varphi_2^0 = \varphi_2(x_0, t) \tag{4.18}$$

$$\beta = \varphi_2^0 \exp\left(\int_{x_0}^{x} v\,dx\right) \tag{4.19}$$

$$\hat{B} = \int_{x_0}^{x} \beta^2\,dx \tag{4.20}$$

$$\hat{B}' = \int_{x_0}^{x} \beta^{-2}\,dx \tag{4.21}$$

Then,

$$a = \beta^4(a_0 - b_0\hat{B}') \tag{4.22}$$

$$b = b_0\beta^2 \tag{4.23}$$

$$c = \beta^2(a_0\hat{B} - b_0\hat{B}\hat{B}' + c_0 - d_0\hat{B}) \tag{4.24}$$

$$d = b_0\hat{B} + d_0 \tag{4.25}$$

where

$$a_0 = a_1 + b_1\hat{A} \tag{4.26}$$

$$b_0 = b_1 \tag{4.27}$$

$$c_0 = -a_1\hat{A}' - b_1\hat{A}\hat{A}' + c_1 + d_1\hat{A} \tag{4.28}$$

$$d_0 = -b_1\hat{A}' + d_1 \tag{4.29}$$

$$\hat{A} = \int_{t_0}^{t} (\varphi_2^0)^{-2}\hat{C}[u(x_0, t), t]\,dt \tag{4.30}$$

$$\hat{A}' = \int_{t_0}^{t} (\varphi_2^0)^{2}\hat{C}[u'(x_0, t), t]\,dt \tag{4.31}$$

and a_1, b_1, c_1, and d_1 are some arbitrary constants satisfying the following condition:

$$a_1d_1 - b_1c_1 = 1 \tag{4.32}$$

Let

$$G = G_3^{-1}G_2G_1 \tag{4.33}$$

where G_1, G_2, and G_3 are defined in (2.12), (4.17), and (4.12), respectively. Thus, by (2.13), (4.16), and (4.11), (4.33) is a GT

$$G\colon\quad \Psi \to \Psi' = G\Psi \tag{4.34}$$

which transforms the AKNS systems (2.1) into (4.1).

5. η-DEPENDENT MODIFIED GSGE AND ITS BÄCKLUND TRANSFORMATION

We now apply the GT (4.34) to derive a transformation for the quantity ψ_2/ψ_1 in the BT (3.1).

Denote

$$w=\psi_2/\psi_1, \qquad w'=\psi_2'/\psi_1' \tag{5.1}$$

Then the BT (3.1) can be expressed in terms of w:

$$z'=z+4\tan^{-1} w \tag{5.2}$$

Using (4.13), (4.16), (4.17), (3.15), (4.19), and (2.14), we have

$$\begin{aligned}\psi_1'+i\psi_2'=\varphi_2'&=\frac{1}{\beta^2}\{-c\varphi_1-[(\eta+v)c-d]\varphi_2\}\\&=\frac{\varphi_2}{\beta^2}\left[-c\frac{\varphi_1}{\varphi_2}-(\eta+v)c+d\right]\\&=\varphi_2^{-1}[c(\eta+v)-(\eta+v)c+d]\\&=|\varphi_2|^{-1}(\psi_1-i\psi_2)d\end{aligned} \tag{5.3}$$

Let μ and ν be the real and imaginary parts of the complex function d defined in (4.25), respectively:

$$d=\mu+i\nu \tag{5.4}$$

Substituting (5.4) into (5.3), we get

$$\psi_1'+i\psi_2'=|\varphi_2|^{-2}[(\psi_1\mu+\psi_2\nu)+i(\psi_1\nu-\psi_2\mu)] \tag{5.5}$$

Denote

$$\hat{w}=\nu/\mu \tag{5.6}$$

Then by (5.1), (5.5), and (5.6), we obtain

$$w'=\frac{\psi_1\nu-\psi_2\mu}{\psi_2\mu+\psi_1\nu}=\frac{\hat{w}-w}{1+\hat{w}w} \tag{5.7}$$

(5.7) is the transformation formula for the function w defined in (5.1).

The function w is in fact a solution of another evolution equation. We now derive this equation. Taking the derivative with respect to x in the first equality of (5.1) and using (2.1)-(2.7), we get

$$w_x=-2\eta w-q(1+w^2) \tag{5.8}$$

Solving for q from (5.8) gives

$$q=-(w_x+2\eta w)/(1+w^2) \tag{5.9}$$

Again taking the derivative with respect to t in the first equality of (5.1) and using (2.1)-(2.7), we get

$$w_t = -2Aw - Bw^2 + C \tag{5.10}$$

Substituting (2.37)-(2.39) into (5.10) and using (2.34) and (2.30) gives

$$w_t = \frac{1}{2} \sum_{j=0}^{n} [(4wD^{-1}q + w^2 - 1)\eta + 2(w^2+1)D^{-1}G] \times G^{n-j}\eta^{-2(n-j+1)}q_t \tag{5.11}$$

where the function q by (5.9), is a functional of w. Therefore (5.11) is a nonlinear evolution equation about w with η as a parameter, and (5.7) is a BT for it. We call (5.11) the η-dependent modified GSGE (η-mGSGE). Note that when $n = 0$, (2.34), (2.30), and (2.25) imply

$$Gq_t = 0 \tag{5.12}$$

Therefore, for $n = 0$, (5.11) together with (5.12) gives

$$w_t = \frac{1}{2}(4wD^{-1}q + w^2 - 1)\eta^{-1}q_t \tag{5.13}$$

The right-hand side of this equation can be explicitly expressed in terms of w. By (2.30) and (1.2) we have

$$w_t = -\frac{1}{4}\eta^{-1}[2w \cos z + (1 - w^2) \sin z] \tag{5.14}$$

Using the identities

$$2w = (1 + w^2) \sin 2(\tan^{-1} w) \tag{5.15}$$

$$1 - w^2 = (1 + w^2) \cos 2(\tan^{-1} w) \tag{5.16}$$

we find that (5.14) becomes

$$w_t = -\frac{1}{4}\eta^{-1}(1 + w^2) \sin(z + 2 \tan^{-1} w) \tag{5.17}$$

On the other hand, (2.30) and (5.9) give

$$z = -2 \tan^{-1} w - 4\eta \int \frac{w}{1 + w^2}\, dx \tag{5.18}$$

Then, by substituting (5.18) into (5.17), we get the final form of the η-mGSGE (5.11) for $n = 0$ as follows:

$$w_t = \frac{1}{4}\eta^{-1}(1 + w^2) \sin 4\eta \int \frac{w}{1 + w^2}\, dx \tag{5.19}$$

6. SUMMARY AND EXAMPLE

By the BTs (5.2) and (5.7) and the GT (4.34) we can now start from a known solution z_1 of the GSGE (2.34) to obtain a hierarchy of solutions of that equation,

$$z_1, z_2, z_3, \ldots, z_k, \ldots \tag{6.1}$$

a hierarchy of solutions of the corresponding AKNS system (2.1),

$$\Psi_1, \Psi_2, \Psi_3, \ldots, \Psi_k, \ldots \tag{6.2}$$

and a hierarchy of solutions of the η-mGSGE (5.11),

$$w_1, w_2, w_3, \ldots, w_k, \ldots \tag{6.3}$$

without solving any differential equation except for Ψ_1 in the following manner:

$$\begin{array}{cccccccc} z_1 & \longrightarrow & z_2 & \longrightarrow & z_3 & \longrightarrow & z_4 & \longrightarrow \\ \downarrow & \nearrow w_1 \nearrow & \downarrow & \nearrow w_2 \nearrow & \downarrow & \nearrow w_3 \nearrow & \downarrow & \\ \Psi_1 & \longrightarrow & \Psi_2 & \longrightarrow & \Psi_3 & \longrightarrow & & \end{array} \tag{6.4}$$

Example. We now use the above results to obtain solutions of the GSGE (2.34) for $n=0$, that is, the SGE (1.2):

$$z_{xt} = \sin z \tag{6.5}$$

from a known solution up to the third generation.

Put $n=0$ in (2.37)-(2.39); we have

$$A = \tfrac{1}{4}\eta^{-1} \cos z \tag{6.6}$$

$$B = -\tfrac{1}{4}\eta^{-1} \sin z \tag{6.7}$$

$$C = -\tfrac{1}{4}\eta^{-1} \sin z \tag{6.8}$$

Substituting (2.30) and (6.6)-(6.8) into (2.1)-(2.7), we get the corresponding AKNS system,

$$d\Psi = \begin{pmatrix} \eta & \frac{1}{2}z_x \\ -\frac{1}{2}z_x & -\eta \end{pmatrix} \Psi\, dx + \begin{pmatrix} \frac{1}{4}\eta^{-1}\cos z & -\frac{1}{4}\eta^{-1}\sin z \\ -\frac{1}{4}\eta^{-1}\sin z & \frac{1}{4}\eta^{-1}\cos z \end{pmatrix} \Psi\, dt \tag{6.9}$$

Equation (6.5) possesses a trivial solution

$$z_1 = 2m\pi \tag{6.10}$$

where m is an arbitrary integer. Substituting (6.10) into (6.9), we have

$$d\Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi\, d\xi, \qquad \xi = \eta x + \frac{1}{4\eta} t \tag{6.11}$$

Solving (6.11), we get the first generation of solution of the AKNS system (6.9),

$$\Psi=\begin{pmatrix} e^{\xi} & 0 \\ 0 & e^{-\xi} \end{pmatrix}\Psi_0 \tag{6.12}$$

where Ψ_0 is a constant column vector. Taking

$$\Psi_0=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.13}$$

then (6.12) gives

$$\Psi=\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}=\begin{pmatrix} e^{\xi} \\ e^{-\xi} \end{pmatrix} \tag{6.14}$$

Substituting (6.14) into (5.1), we get

$$w_1=\frac{\psi_2}{\psi_1}=e^{-2\xi} \tag{6.15}$$

This is the first generation of solutions of the η-mSGE (5.19). Now, by the BT (5.2) of the GSGE (2.34) and (6.15), we get the second generation of solutions of (2.34),

$$z_2=2m\pi+4\tan^{-1}\exp(-2\xi) \tag{6.16}$$

To obtain the subsequent generation of solutions, we use formula (5.7). We must calculate the $\hat{w}$ first, or by (5.6) and (5.4), we have to calculate the complex function d defined in (4.25). By (4.25), (4.27), and (4.29) we have

$$d=b_1(\hat{B}-\hat{A}')+d_1 \tag{6.17}$$

Referring to (4.20), (4.19), (4.18), (3.15), and (2.14), we get

$$\hat{B}=\int_{x_0}^{x}(\psi_1+i\psi_2)^2\,dx \tag{6.18}$$

Inserting (6.14) into (6.18) gives

$$\hat{B}=\eta^{-1}(\cosh 2\xi-\cosh 2\xi_0)+2i(x-x_0) \tag{6.19}$$

where

$$\xi_0=\eta x_0+\tfrac{1}{4}\eta^{-1}t \tag{6.20}$$

The function $\hat{A}'$ in (6.17) is defined by (4.31). By (4.15), (3.41), and (6.6), we have

$$\hat{C}[u'(x_0,t),t]=\hat{C}'(x=x_0)=-\tfrac{1}{4}\eta^{-2}\exp(-iz_{20}) \tag{6.21}$$

where z_{20} is the value of function (6.16) at the point x_0. Substituting (6.12) into (4.31) and using (4.18), after simplification, we get

$$\hat{A}' = -\eta^{-1}[\cosh 2\xi_0 - \cosh 2\xi_{00} - 2i(\xi_0 - \xi_{00})] \tag{6.22}$$

where

$$\xi_{00} = \eta x_0 + \tfrac{1}{4}\eta^{-1} t_0 \tag{6.23}$$

Applying (6.19) and (6.22) to (6.17) gives

$$d = b_1 \eta^{-1}\{\cosh 2\xi - \cosh 2\xi_{00} + 2i[\eta(x - x_0) - \tfrac{1}{4}\eta^{-1}(t - t_0)]\} + d_1 \tag{6.24}$$

We choose the constants b_1 and d_1 to satisfy the relation

$$d_1 = b_1 \eta^{-1} \cosh 2\xi_{00} \tag{6.25}$$

and make the corresponding choice in (4.32) simultaneously for a_1 and c_1. Thus, by (6.25), (6.24), (5.4), and (5.6), we obtain

$$\hat{w} = \frac{2[\eta(x - x_0) - \tfrac{1}{4}\eta^{-1}(t - t_0)]}{\cosh 2\xi} \tag{6.26}$$

Note that (5.7) can be rewritten in the following equivalent form:

$$w' = \tan(\tan^{-1} \hat{w} - \tan^{-1} w) \tag{6.27}$$

Substituting (6.15) and (6.26) into (6.27), we get the second generation of solutions for the η-mSGE (5.19),

$$w_2 = \tan\left\{\tan^{-1} \frac{2[\eta(x - x_0) - \tfrac{1}{4}\eta^{-1}(t - t_0)]}{\cosh 2\xi} - \tan^{-1} \exp(-2\xi)\right\} \tag{6.28}$$

Applying (6.28) to (5.2), we arrive at the desired third generation of solutions of the SGE (6.5),

$$z_3 = 2m\pi + 4 \tan^{-1} \frac{2[\eta(x - x_0) - \tfrac{1}{4}\eta^{-1}(t - t_0)]}{\cosh 2\xi} \tag{6.29}$$

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